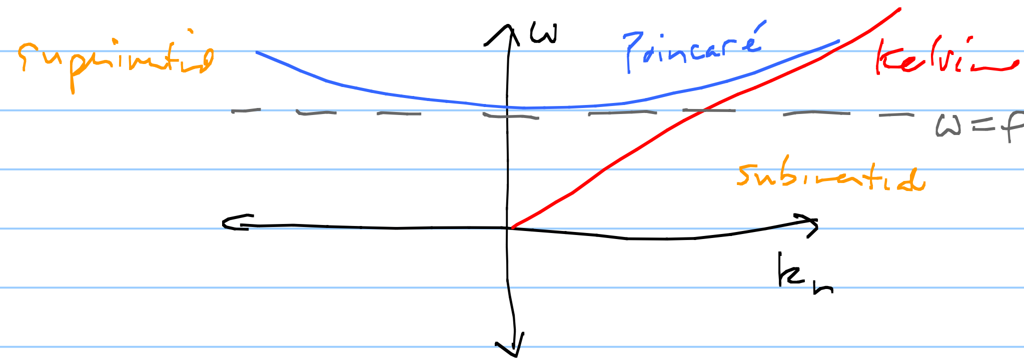
Lecture 9

Note Title 5/12/2009

Last time we talked about two types of waves that carried no PV: Poincaré waves and Kelvin waves:



You might be getting the impression that all waves in the ocean do not have a PV signature. Today we will talk about a type of waves whose very existence, its restoring force depends on the PV, these waves are known as Rossby or sometimes planetary waves and they play an important role in the adjustment of the ocean circulation.

Rossby waves arise from conservation of PV in a medium that has a gradient in the PV. The PV:

$$q = \frac{f + \zeta}{H + \eta}$$

Which we can split into a "background" component, and a deviation from that background:

$$q = \bar{q} + q'$$

The background PV field for this example is associate with the Coriolis parameter and the mean thickness of the water column that do not depend on the flow:

$$\bar{q} = \frac{f}{H}$$

$\bar{q}$ can vary in space, due to the N-S variation in f and/or variations in the water column

$$\bar{q}(x,y) = \frac{f(y)}{H(x,y)}$$

The statement of PV conservation is:

$$\frac{Dq}{Dt} = 0 \qquad \text{(No friction)}$$

$$\frac{Dq'}{Dt} + \frac{D\bar{q}}{Dt} = 0$$

$\bar{q}$ is independent of time but varies in space so that

$$\frac{D\bar{q}}{Dt} = \vec{u}\cdot\nabla\bar{q}$$

Thus PV conservation implies:

$$\frac{Dq'}{Dt} = -\vec{u}\cdot\nabla\bar{q}$$

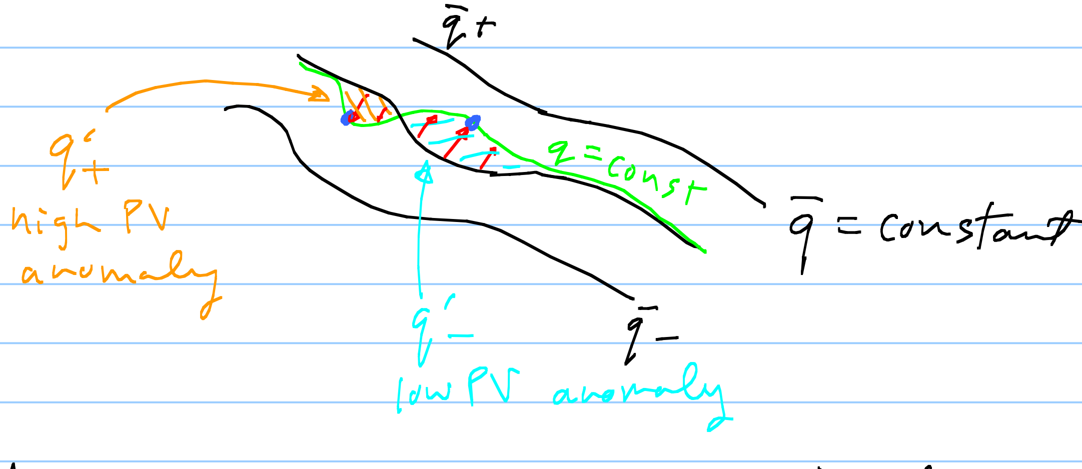
Noting that $\frac{D\vec{X}}{Dt} \equiv \vec{u}$ $\quad \vec{X}$ is the displacement of fluid parcels

$$\frac{Dq'}{Dt} = -\frac{D\vec{X}}{Dt} \cdot \nabla \bar{q}$$

This can be integrated in time following a fluid parcel yielding:

$$q' = -\vec{X} \cdot \nabla \bar{q}$$

$\Rightarrow$ Displacements of fluid parcels when they have a component $\parallel$ to the background / mean PV gradient will induce PV anomalies:



As we saw in the geostrophic adjustment problem, PV anomalies have associated with them flow anomalies, as a consequence of PV invertibility:

$$q' \xrightarrow[\text{through PV invertibility}]{} \vec{u}$$

Now these flow anomalies will result in a change in the particle's displacement since

$$\frac{D\vec{X}}{Dt} = \vec{u}$$

the change in the displacement field will modify the PV anomaly and this feedback can give rise to a restoring force that leads to wavelike phenomena, these waves are Rossby waves.

Lets look at a concrete example of this for the case of the PV gradient caused by the variation of the Coriolis parameter with latitude.

Consider a uniform depth layer of water of thickness H on a β-plane

$$f = f_0 + \beta y$$

The background PV field is

$$\bar{q} = \frac{f + \beta y}{H} \rightarrow \nabla\bar{q} = \frac{\beta}{H}\hat{j}$$

Lets perturb this system by a weak flow field with velocity u, v and free surface displacement η.

By weak we specifically mean that the Rossby number of the flow is small, that the displacements in the free surface are much smaller than H, and that north-south displacements δ_y are small enough so that $\beta \delta_y << f$. In this limit the PV anomaly becomes:

$$q' = \frac{\zeta}{H} - \frac{\eta f}{H^2}$$

Let's further assume that the flow is completely geostrophic:

$$u = -\frac{g}{f}\frac{\partial \eta}{\partial y} \qquad v = \frac{g}{f}\frac{\partial \eta}{\partial x}$$

Thus we can introduce a geostrophic streamfunction $u = -\partial\psi/\partial y$, $v = \partial\psi/\partial x$

$$\psi = \frac{g}{f}\eta$$

Then

$$q' = \frac{\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)}{H} - \frac{\eta f}{H^2}$$

$$= \frac{\nabla^2 \psi}{H} - \frac{\psi f^2}{gH\,H}$$

$$\boxed{q' = \frac{\nabla^2 \psi}{H} - \frac{\psi}{L_r^2\, H}} \qquad L_r = \frac{\sqrt{gH}}{f}$$

So then the PV equation becomes:

$$\frac{Dq'}{Dt} = -\vec{u}\cdot\nabla\bar{q} = -\frac{v\beta}{H} = -\frac{\partial\psi}{\partial x}\frac{\beta}{H}$$

$$\frac{1}{H}\frac{D}{Dt}\left(\nabla^2\psi - \frac{\psi}{L_r^2}\right) = -\frac{\partial\psi}{\partial x}\frac{\beta}{H}$$

$$\frac{D}{Dt}\left(\nabla^2\psi - \frac{\psi}{L_r^2}\right) = -\beta\frac{\partial\psi}{\partial x}$$

Note that

$$\vec{u}\cdot\nabla q' = u\frac{\partial q'}{\partial x} + v\frac{\partial q'}{\partial y}$$

$$= -\frac{\partial\psi}{\partial y}\frac{\partial q'}{\partial x} + \frac{\partial\psi}{\partial x}\frac{\partial q'}{\partial y}$$

$$= J(\psi, q')$$

Where J is the Jacobian operator:

$$J(A,B) = -\frac{\partial A}{\partial y}\frac{\partial B}{\partial x} + \frac{\partial B}{\partial y}\frac{\partial A}{\partial x}$$

Note that $J(A,A) = 0$

Finally the PV equation becomes:

$$\boxed{\frac{\partial}{\partial t}\left(\nabla^2\psi - \frac{\psi}{L_r^2}\right) + J\left(\psi, \nabla^2\psi - \frac{\psi}{L_r^2}\right) + \beta\frac{\partial\psi}{\partial x} = 0}$$

This is the QUASI-GEOSTROPHIC PV EQUATION, and the quantity

$$\nabla^2 \psi - \frac{\psi}{L_r^2}$$

is know as the QG - PV (but note that it has units of vorticity).

We derived this equation by essentially restricting the flow to be geostrophic. I just basically told you that the flow is geostrophic, I didn't prove to you under what conditions is this assumption valid. One can perform a rigorous regular perturbation expansion in Rossby number of the shallow water equations and show that for small but non-zero Ro, the QG equations are valid! If we have time, I will show you how this is done!

Let's first show that PV conservation yields wavelike phenomena. We can do this by looking for plane wave solutions to the QG-equation:

$$\psi = \mathrm{Re}\left(\psi_0 e^{i(\vec{k}\cdot\vec{x} - \omega t)}\right)$$

where $\vec{k} = (k, l)$ $\quad \omega > 0$

Note that $\nabla^2\psi = -(k^2+l^2)\psi$

So that the advection term:

$$J(\psi, \nabla^2\psi - \psi/L_r^2) = -\left[(k^2+l^2)+1/L_r^2\right]\cancelto{0}{J(\psi,\psi)}$$

Thus the PV equation becomes:

$$-i\omega\left[-(k^2+l^2) - \frac{1}{L_r^2}\right] + ik\beta = 0$$

$$\boxed{\omega = \frac{-\beta k}{\left[(k^2+l^2)+(1/L_r)^2\right]}}$$

DISPERSION RELATION FOR ROSSBY WAVES

For the time being, let's first consider a streamfunction that only varies in the x-direction, i.e. there is only a meridional flow and $\vec{k} = (k, 0)$. Then

$$\omega = -\frac{\beta k}{k^2 + (1/L_r)^2}$$

This can be non-dimensionalized in terms of f and rewritten as:

$$\frac{\omega}{f} = -\left(\frac{\beta L_r}{f}\right)\frac{(kL_r)}{\left[(kL_r)^2+1\right]} \qquad \hat{k} = L_r k$$

$$\frac{\partial}{\partial \hat{k}}\left(\frac{\hat{k}}{\hat{k}^2+1}\right) = \frac{1}{\hat{k}^2+1} - \frac{2\hat{k}^2}{(\hat{k}^2+1)^2} = 0$$

$$\hat{k}^2 + 1 - 2\hat{k}^2 = 0 \qquad k^2 = 1 \qquad k = \pm 1$$

Frequency is maximum at $k = -1/L_r$ and is equal to:

$$\left.\frac{\omega}{f}\right|_{max} = + \frac{\beta L_r}{2f}$$

Recall that $\beta = \frac{2\Omega \cos(\text{latitude})}{R_e}$

which at mid-latitudes is about $\beta \simeq 2\times10^{-11}\,m^{-1}s^{-1}$
$L_r = \sqrt{gH}/f$, with $g = 9.8\ m/s^2$
$f \simeq 1\times10^{-4}\,s^{-1}$ (a typical mid-latitude value)
and $H \simeq 4000\ m$, then

$$L_r \simeq \frac{\sqrt{9.8 \times 4000}}{10^{-4}} \simeq \frac{200\ m/s}{10^{-4}\,s^{-1}}$$

$$\simeq 2000\ km$$

$$\beta L_r = (2\times10^{-11}\,m^{-1}s^{-1})(2\times10^{6}\,m) = 4\times10^{-5}\,s^{-1}$$

$$\left.\frac{\omega}{f}\right|_{max} = \frac{4\times10^{-5}\,s^{-1}}{2\times10^{-4}\,s^{-1}} = 0.2 < 1$$

$\Rightarrow$ Rossby waves are subinertial.

Another key property of Rossby waves is that their phase velocity in the x-direction:

$$C_p^x = \frac{\omega}{k} = \frac{-\beta}{[k^2 + (1/L_r)^2]} < 0$$

is always less than zero, i.e.

$$\boxed{\text{The phase lines of Rossby waves (i.e. streamlines and hence lines of constant pressure or free surface displacement) always propagate to the WEST}}$$

This can be seen in observations of the sea surface height anomalies measured using satellite altimetry.
→ show animation in PPT

What is the physics behind this westward phase propagation?

The answer to this question depends on the size of the wavelength of the wave relative to the Rossby radius.

When the wavelength of the wave λ is small compared to the Rossby radius $\lambda << L_r$ or equivalently:

$$k L_r >> 1$$ short wave limit

the PV anomaly :

$$q' = \underbrace{\frac{\zeta}{H}}_{\text{vorticity}} - \underbrace{\frac{\eta f}{H^2}}_{\text{stretching}}$$

is primarily associate with the vorticity as opposed to thickness anomalie (i.e. the so called stretching term.)

Showing this :

$$\left|\frac{\text{vorticity}}{\text{stretching}}\right| = \left|\frac{\nabla^2\psi / H}{-\psi / L_r^2 H}\right| = \left|\frac{\nabla^2\psi L_r^2}{\psi}\right|$$

For wavelike solutions $\nabla^2\psi = -k^2\psi$

$$\left|\frac{\text{vorticity}}{\text{stretching}}\right| \sim k^2 L_r^2$$

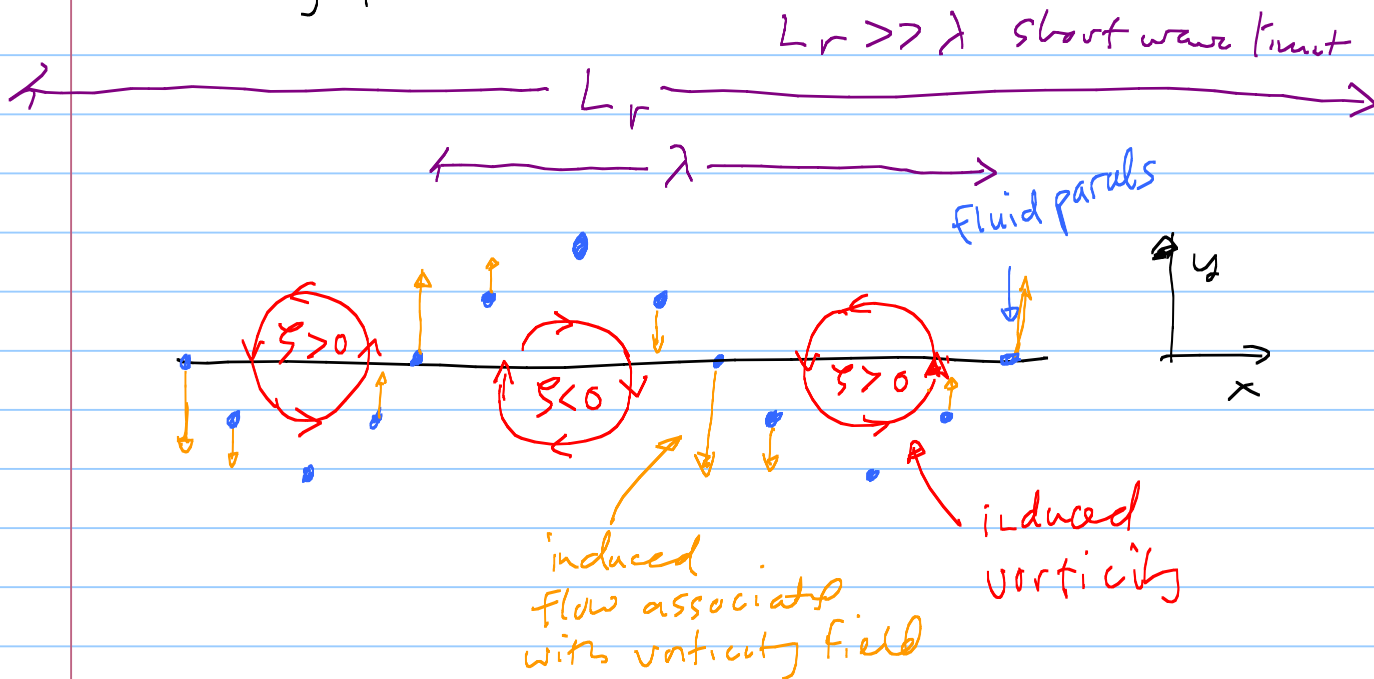
Thus in the shortwave limit

$$q' \approx \frac{\zeta}{H}$$

And thus the total PV :

$$q = \bar{q} + q' \approx \frac{f+\zeta}{H}$$

Consequently a line of particles is displaced from a latitude circle; to conserve PV they will induce a vorticity field:



The induced flow associated with the vorticity field will displace the fluid parcels in such a way as to cause the whole pattern to shift to the left, i.e. towards the west, thus the PV anomaly and wavephase will propagate to the west.

What the case when the wavelength of the wave is much larger than the Rossby radius: $\lambda >> L_r$

$$kL_r << 1$$ long wave limit

In this case

$$\left|\frac{\text{vorticity}}{\text{stretching}}\right| \sim (k L_r)^2 \ll 1$$

so that the PV anomaly is approximately

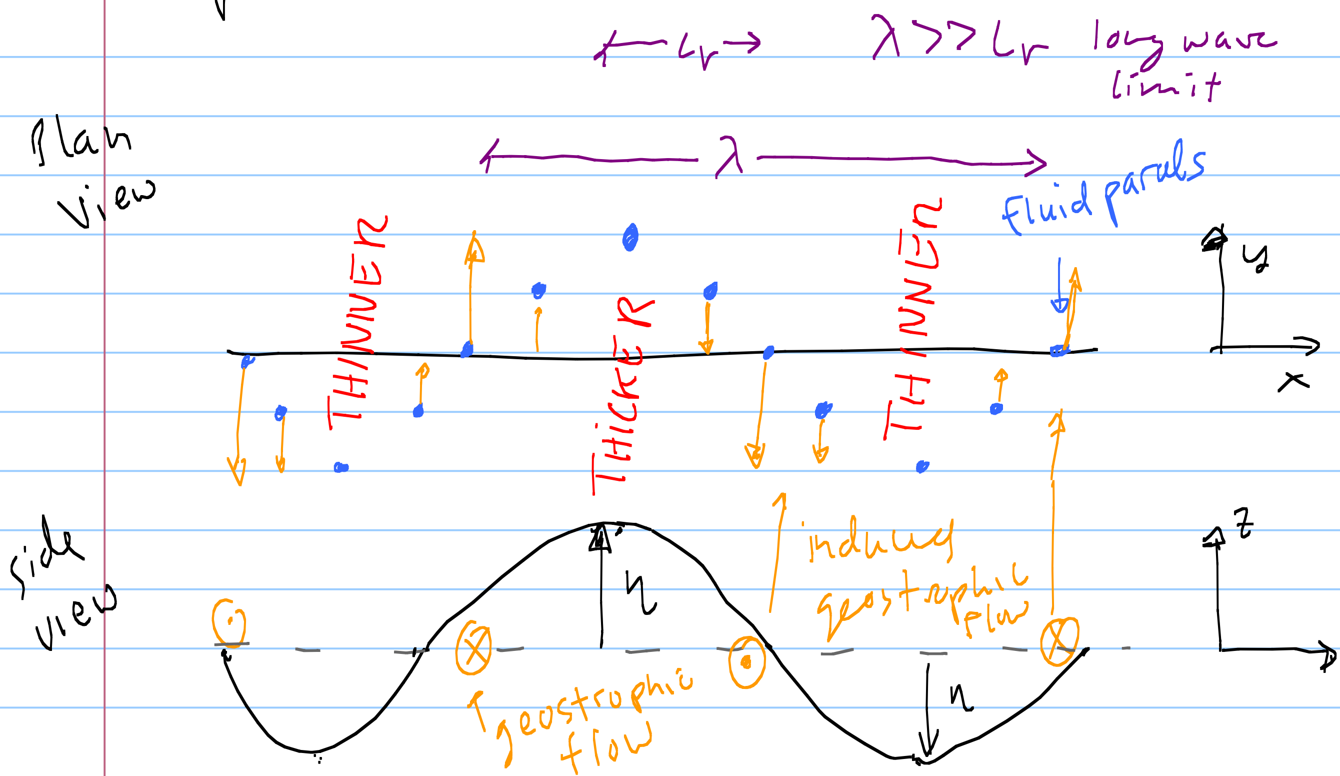
$$q' \simeq -\frac{f\eta}{H^2}$$

And the total PV is

$$q = \bar{q} + q' \simeq \frac{f}{H+\eta}$$

i.e. the fluid conserves its PV through changing its thickness not vorticity.

Lets again look at the fluid parcel displacements:



This explains the physics behind the westward PHASE velocity of Rossby waves. What about their GROUP velocity?

Recall that the group velocity of any type of wave is:

$$\vec{c}_g = \nabla_k \omega$$

where $\nabla_k = \frac{\partial}{\partial k}\hat{i} + \frac{\partial}{\partial l}\hat{j} + \frac{\partial}{\partial m}\hat{k}$

and (k, l, m) are the components of the wavevector $\vec{k}$.

Lets calculate the group velocity of Rossby waves that only vary in the x- direction. The dispersion relation is from above:

$$\omega = \frac{-\beta k}{k^2 + (1/Lr)^2}$$

$$c_g^x = \frac{\partial \omega}{\partial k} = \frac{-\beta}{k^2 + (1/Lr)^2} + \frac{2\beta k^2}{[k^2 + (1/Lr)^2]^2}$$

$$= \frac{-\beta[k^2 + (1/Lr)^2] + 2\beta k^2}{[k^2 + (1/Lr)^2]^2}$$

$$= -\beta \frac{[(1/Lr^2) - k^2]}{[k^2 + (1/Lr)^2]^2}$$

Which you will notice is :

$$c_g^x = 0 \quad \text{when} \quad k = 1/L_r$$

$$c_g^x > 0 \text{ , i.e eastward } \quad k > 1/L_r$$

$$c_g^x < 0 \text{ , i.e westward } \quad k < 1/L_r$$

Recall that the group velocity tells you the direction and speed that the energy of the waves is trasmitted. The above result shows that:

LONG WAVES with $kL_r < 1$
transmit energy westward

SHORT WAVES with $kL_r > 1$
transmit energy eastward

Also notice that the magnitude of the group velocity differs dramatically for long and short waves:

For $kL_r \ll 1$

$$|c_g^x| \approx \beta L_r^2 = |c_p^x|$$

$\Rightarrow$ The group and phase speeds are equal for long waves, i.e. they are non-dispersive.

For short waves $kL_r >> 1$

$$|c_g^x| \simeq \frac{\beta}{k^2}$$

$\Rightarrow$ which decreases with increasing wave number.

So comparing the magnitudes of the group velocity of Rossby waves in the long and short wave limits:

$$\frac{kL_r << 1 \; |c_g^x|}{kL_r >> 1 \; |c_g^x|} \sim \underset{\substack{\uparrow \\ \text{evaluated} \\ \text{for } kL_r >> 1}}{L_r^2 k^2} >> 1$$

$\Rightarrow$ Long waves transmit energy much more rapidly than short waves

This has very important implications for the spin-up of the large scale circulation of the ocean by the winds.

For baroclinic motions in the ocean the key length scale is the baroclinic Rossby radius of deformation

$$L_r^{bc} \simeq 50 - 100 \text{ km}$$

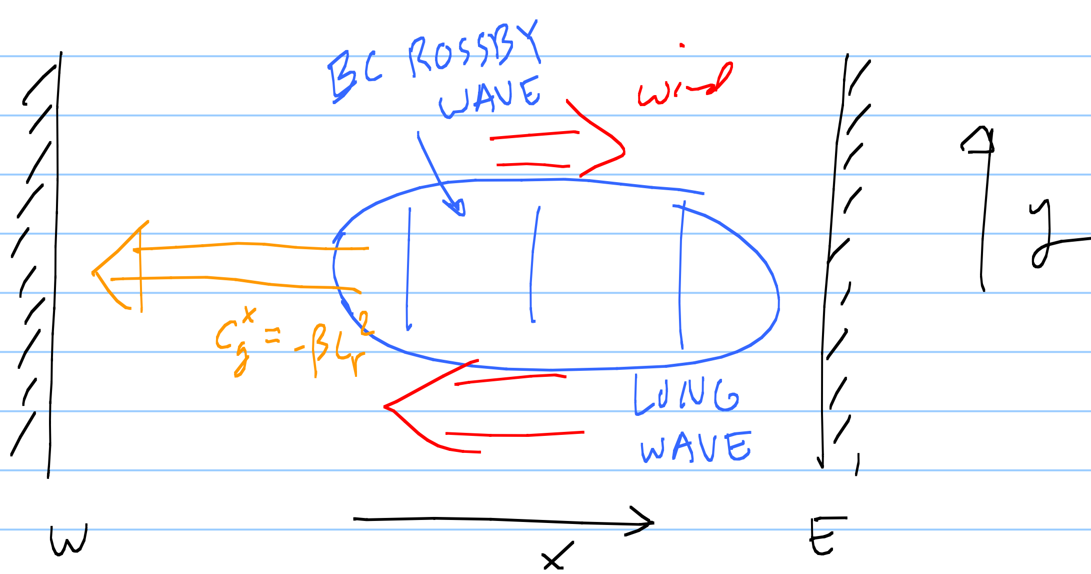
at midlatitudes.

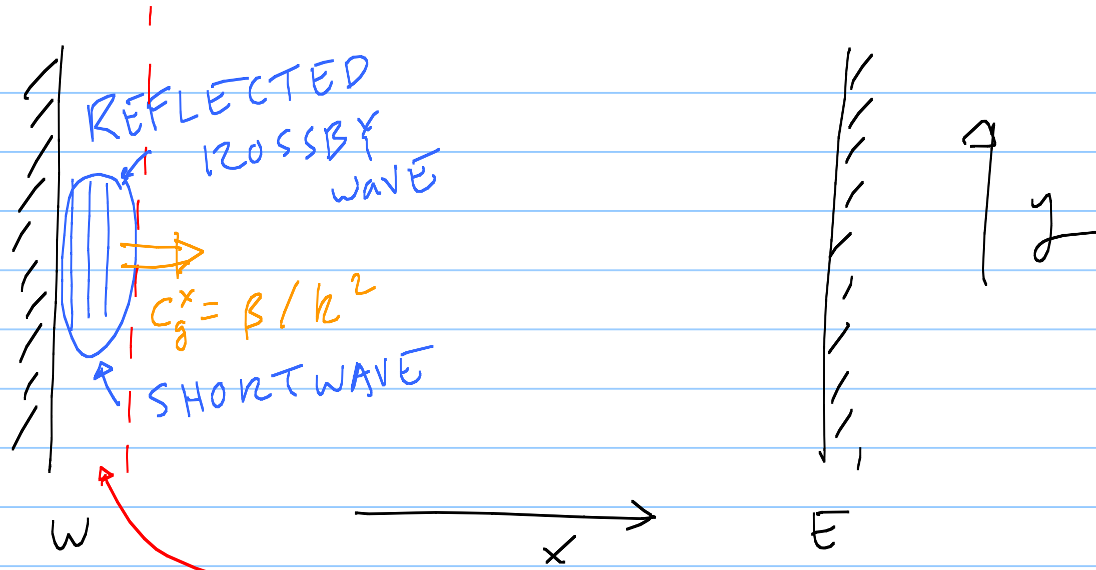
Baroclinic Rossby waves with $kL_r^{bc} \ll 1$ are LONG WAVES and have a westward group velocity.

Baroclinic Rossby waves w/ $kL_r^{bc} \gg 1$ as SHORT WAVES and have an eastward group velocity.

Similar to the barotropic Rossby waves discussed above, the eastward group velocity of SHORT baroclinic waves is much smaller than the westward group velocity of the LONG waves.

The wind forcing of the ocean has length scales of order 1000 km, and if it is time variable will generate Rossby waves with length scales much larger than the Rossby radius, i.e. it will generate LONG waves.





Near the western boundary, energy is being fluxed in from the east at a rate proportional to $c_g^x = -\beta L_r^2$, the reflected SHORT wave Rossby waves flux energy out of this region at a rate proportional to $c_g^x = \beta k^2$
Since

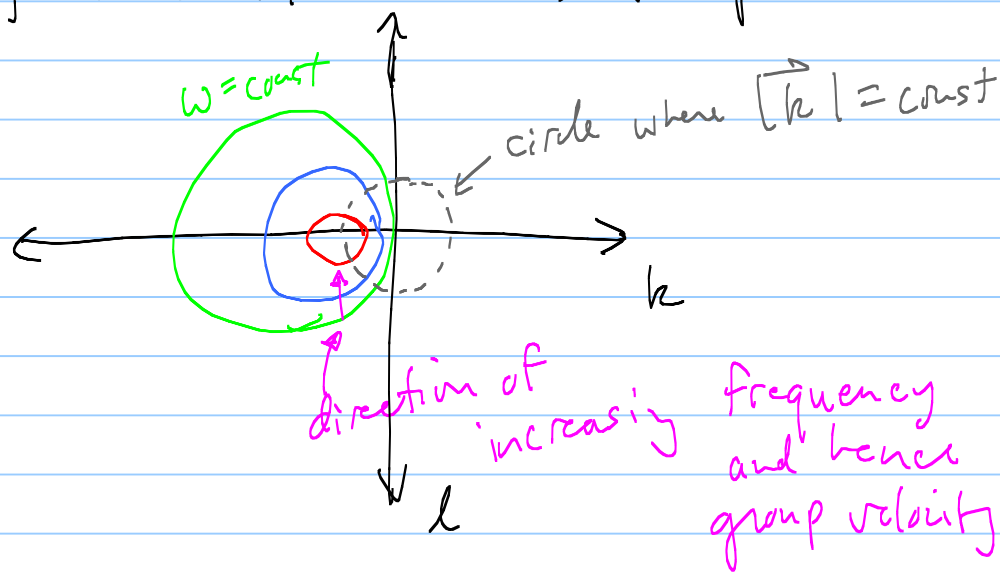
$$|-\beta L_r^2| >> \beta / k^2$$

More energy is fluxed into the region than is fluxed out, i.e. there is a convergence of energy near the western boundary. This explains why the most energetic flows in the oceans are found on the western side of ocean basin, a phenomenon called WESTERN INTENSIFICATION.

So far we have focused on 1-D Rossby waves that have no variation in the y-direction. When we allow for a y-component to the wavevector, the dispersion relation is:

$$\omega = -\frac{\beta k}{k^2 + l^2 + (1/L_R)^2}$$

Plotting this in the l-k plane:

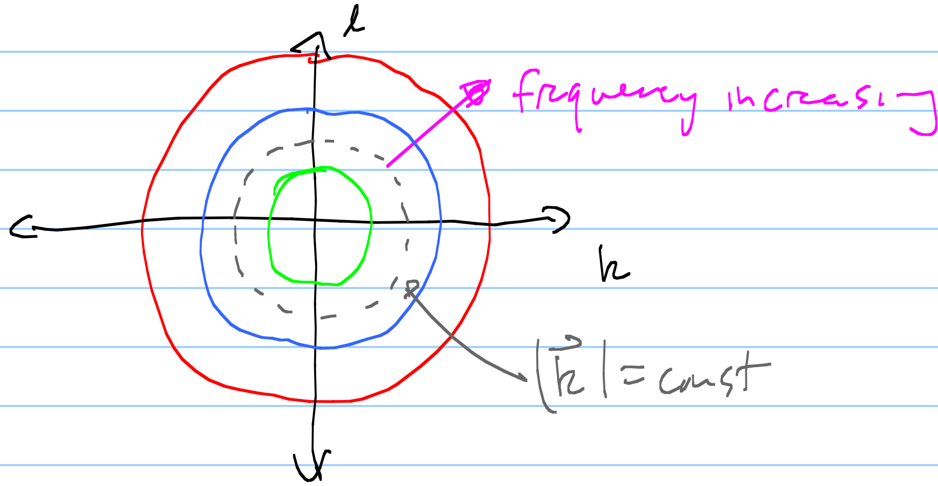


The contours of constant frequency are circle centered on the l axis but displaced to the left of k = 0.

This displacement causes the waves to be ANISOTROPIC ie Their frequency, phase and group velocity depends on the direction of the wave vector not just its magnitude. For example a circle centered about the origin $|\vec{k}|$ = constant, but the frequency changes depending on the direction of the wavevector.

Contrast this to the dispersion relation of 2D Poincaré waves:

$$\omega = \sqrt{f^2 + gH(k^2+l^2)}$$



The frequency, phase and group velocities are constant for a fixed magnitude of the wavevector, i.e. these waves are ISOTROPIC.